Newton-Raphson Method, Secant Method, & Method of False Position

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Newton's (or the Newton-Raphson) Method

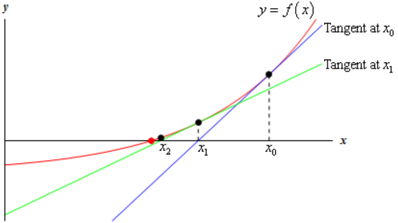
Newton's (or the Newton-Raphson) method is one of the most powerful and well-known numerical methods for solving a root-finding problem.

There are many ways of introducing Newton's method.

First we explain the algorithm graphically.

The figure illustrates how the approximations are obtained using successive tangents. Starting with the initial approximation x_0, the approximation x_1 is the x-intercept of the tangent line to the graph of f at $(x_0, f(x_0))$.

The approximation x_2 is the x-intercept of the tangent line to the graph of f at $(x_1, f(x_1))$ and so on.



We introduce Newton's method by using Taylor polynomials.

Suppose $f \in C^2[a, b]$. Let $x_0 \in [a, b]$ be an approximation to the solution α of $f(x) = 0$ such that $f'(x_0) \neq 0$ and $|x - x_0|$ is "small".

Consider the first Taylor polynomial for $f(x)$ expanded about x_0, and evaluated at $x = \alpha$,

$$f(\alpha) = f(x_0) + (\alpha - x_0)f'(x_0) + \frac{(\alpha - x_0)^2}{2} f''(\xi(\alpha)),$$

where $\xi(\alpha)$ lies between α and x_0.

Since $f(\alpha) = 0$, this equation gives

$$0 = f(x_0) + (\alpha - x_0)f'(x_0) + \frac{(\alpha - x_0)^2}{2} f''(\xi(\alpha)).$$

Newton's method is derived by assuming that since $|\alpha - x_0|$ is small, the term involving $(\alpha - x_0)^2$ is much smaller, so

$$0 \approx f(x_0) + (\alpha - x_0)f'(x_0).$$

Solving for x_0 gives

$$\alpha \approx x_1 \equiv x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This sets the stage for Newton's method, which starts with an initial approximation x_0 and generates the sequence $(x_n)_{n=0}^{\infty}$ by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \text{ for } n \geq 1.$$

Newton's method is a functional iteration technique of the form $x_n = g(x_{n-1})$, for which

$$g(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \text{ for } n \geq 1.$$

- Newton's method cannot be continued if $f'(x_{n-1}) = 0$ for some n.
- The method is most effective when f' is bounded away from zero near α.
- **Importance of an accurate initial approximation.** The crucial assumption is that the term involving $(\alpha - x_0)^2$ is, by comparison with $|\alpha - x_0|$, so small that it can be deleted. This will clearly be false unless x_0 is a good approximation to α.
- If x_0 is not sufficiently close to the actual root, there is little reason to suspect that Newton's method will converge to the root.

Theoretical Importance of the Choice of x_0

Theorem (Convergence Theorem for Newton's Method)

Let $f \in C^2[a,b]$. If $\alpha \in [a,b]$ is such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $(x_n)_{n=1}^{\infty}$ converges to α for any initial approximation $x_0 \in [\alpha - \delta, \alpha + \delta]$.

This result is important for the theory of Newton's method, but it is seldom applied in practice since it does not tell us how to determine δ.

In a practical application, an initial approximation is selected, and successive approximations are generated by Newton's method. These will generally either converge quickly to the root, or it will be clear that convergence is unlikely.

Convergence Theorem for Newton's Method states that, under reasonable assumptions, Newton's method converges provided a sufficiently accurate initial approximation is chosen.

It also implies that the constant k that bounds the derivative of g, and, consequently, indicates the speed of convergence of the method, decreases to 0 as the procedure continues.

Major weakness: the need to know the value of the derivative of f at each approximation. Frequently, $f'(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.

Secant Method : Derivation by using Taylor Polynomial

To avoid the derivation in Newton's method, we introduct a slight variation.

By definition,

$$f'(x_{n-1}) = \lim_{x \to x_{n-1}} \frac{f(x) - f(x_{n-1})}{x - x_{n-1}}.$$

Letting $x = x_{n-2}$, we have

$$f'(x_{n-1}) \approx \frac{f(x_{n-2}) - f(x_{n-1})}{x_{n-2} - x_{n-1}} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}.$$

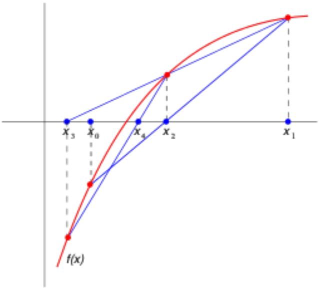
Using this approximation for $f'(x_{n-1})$ in Newton's formula gives

$$x_n = \frac{x_{n-2} f(x_{n-1}) - x_{n-1} f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}.$$

Derivation of Secant Method Graphically

Starting with the two initial approximations x_0 and x_1, the approximation x_2 is the x-intercept of the line joining $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

The approximation x_3 is the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and so on.



Root Bracketing

Each successive pair of approximations in the bisection method brackets a root α of the equation; that is, for each positive integer n, a root lies between a_n and b_n.

This implies that, for each n, the bisection method iterations satisfy

$$|x_n - \alpha| < \frac{1}{2}|a_n - b_n|,$$

which provides an easily calculated error bound for the approximation.

Root bracketing is not guaranteed for either Newton's method or the secant method.

The initial approximations x_0 and x_1 bracket the root, but the pair of approximations x_3 and x_4 fail to do so.

Method of False Position (also called Regula Falsi)

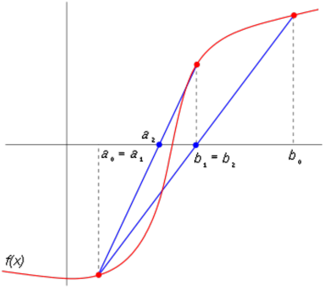
The method of false position generates approximations in the same manner as the secant method, but it includes a test to ensure that the root is always bracketed between successive iterations. It illustrates how bracketing can be incorporated.

First choose initial approximations x_0 and x_1 with $f(x_0).f(x_1) < 0$. Let $a_0 = x_0$ and $b_0 = x_1$. Note that a_0 and b_0 bracket a root.

The approximation x_2 is chosen in the same manner as in the secant method, as the x-intercept of the line joining $(a_0, f(a_0))$ and $(b_0, f(b_0))$.

To decide which secant line to use to compute x_3, we check $f(a_0).f(x_3)$. If this value is negative, then a_0 and x_3 bracket a root. (If not, x_3 and b_0 bracket a root.)

A relabeling of a_0 and x_3 is performed as $a_1 = a_0$ and $b_1 = x_3$. (If not, a relabeling of x_3 and b_0 is performed as $a_1 = x_3$ and $b_1 = b_0$). The relabeling ensures that the root is bracketed between successive iterations.



More Calculations Needed for the "Added Insurance" of the Method of False Position

The added insurance of the method of False Position commonly requires more calculation than the Secant method, just as the simplication that the Secant method provides over Newton's method usually comes at the expense of additional iterations.

References

- Richard L. Burden and J. Douglas Faires, "*Numerical Analysis – Theory ad Applications*", Cengage Learning, New Delhi, 2005.
- Kendall E. Atkinson, "*An Introduction to Numerical Analysis*", John Wiley & Sons, Delhi, 1989.